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On the Principle of Duality in the Geometry of the Sphere.

BY ARTHUR RANUM.

1. In the geometry of the surface of the sphere, or what is practically the same thing, in two-dimensional Riemannian geometry, it has long been known that the principle of duality applies not only to projective theorems, but to metrical ones as well.* But although this principle has been used in spherical trigonometry and in the theory of inscribed polyedra, it does not seem to have been applied to the theory of spherical curves or to the kinematics of the sphere.

In this paper a few of the simpler applications to these subjects will be made. The history of mathematics has shown that before the application of the principle of duality to a given domain is recognized, one of two dual theorems is apt to be discovered long before the other, perhaps because of its closer connection with intuition. In the projective geometry of the plane a case in point is that of Pascal's and Brianchon's theorems. In the metrical geometry of the sphere, similarly, it turns out, as I shall try to show, that certain well-known theorems give rise, by the mere use of duality, to theorems that have an unfamiliar and even startling aspect.

2. The fundamental notion is that of two *polar* configurations, which are defined as two configurations in which every point P of one has a corresponding great circle p of the other, which is its polar with respect to the absolute conic; that is, p is situated at a quadrant's distance from P. We assume the radius of the sphere to be unity. By examining the correspondences that exist between two polar configurations, we easily arrive at all the results of this paper.

To begin with, however, we see that in order to make the duality precise, we must introduce the concept of a *directed great circle*; that is, a circle having a definite direction of description or rotation. By its aid the following dual relations become obvious:

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* For a recent statement of this fact see R. Bonola: "Questioni Riguardanti La Geometria Elementare," edited by F. Enriques (1900), p. 217.

Point of the sphere.	Directed great circle of the sphere.
Pair of diametrically opposite points.	Great circle with both directions.
Any two pairs of diametrically opposite points determine one great circle.	Any two great circles determine one pair of diametrically opposite points.
Spherical *distance* between two points.	*Angle* between two directed great circles.

In a spherical polygon, therefore, the sides are not dual to the *interior* angles; in the special case in which the sides and interior angles are all less than π, and the sides are all directed alike with respect to a point describing the perimeter, the sides are dual to the *exterior* angles.

A circle* is the locus of a point situated at a constant distance from a fixed point called the *center*.	A circle* is the envelope of a directed great circle making a constant angle with a fixed (directed) great circle called the *axis*.

The Duality of Arc and Area.

3. By considering a polygon with an infinite number of sides and passing to the limit, we see that the length of the path, or the linear displacement, of a point moving along a curve from one position P to another Q is equal to the angular displacement of its polar great circle in enveloping the polar curve from the position of the tangent p to that of the tangent q. Here the term angular displacement means the limit of the sum of the infinitesimal angles between successive tangents, which is, of course, not equal, in general, to the angle between the initial and terminal great circles p and q. We can therefore introduce the following duality:

The *linear* displacement of a moving point.	The *angular* displacement of a moving great circle.

4. Since the area of a lune is equal to twice the angle between its bounding great circles, therefore the area swept out by a moving great circle is twice its angular displacement. Here it will be understood that the area swept out by a

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* Cf. Coolidge: "Non-Euclidean Geometry" (1909), p. 131.

moving great circle means the area swept out by that semicircle which extends in the positive direction from the point of tangency with the envelope to the diametrically opposite point. The duality just introduced may now be written as follows:

Twice the length of the *arc* described by a moving point.	The *area* swept out by a moving great circle.

5. The theorem implied by this may be given still another statement, as follows: The area included between a curve C and its opposite curve $\overline{C}$, and bounded by two of their common tangents p and q, is equal to twice the length of the corresponding arc of its polar curve C' bounded by the points P and Q, the poles of p and q, respectively.

Here the area bounded by p and q must be understood to mean the area swept out by the tangent in moving from p to q. If an inflectional tangent r lies between p and q, then it will be convenient, although by no means absolutely necessary, to consider the area swept out between p and r and the area swept out between r and q as being of opposite algebraic signs; if so, then in the polar curve, where the point R is a cusp lying between P and Q, the arc PR and the arc RQ must be given opposite algebraic signs. Hence we have the duality:

Twice the length of the arc of a curve bounded by two of its points.	The area between a curve and its opposite curve and bounded by two of their common tangents.
The curve is assumed to have no *point-discontinuity*; that is, to be the locus of a continuously moving point.	The curve is assumed to have no *tangential discontinuity*; that is, to be the envelope of a continuously turning great circle.

Finally, the area between a curve and its opposite curve can be regarded as the *area exterior* to the curve. Hence:

Twice the length of the arc of a curve between two of its points.	The area exterior to a curve and lying between two of its tangents.

6. Now suppose the curves to be closed. In that case the entire area between C and $\overline{C}$ is twice the length of the entire arc of C'; and since the area of the sphere, which is 4π, is made up of the area between C and $\overline{C}$ and twice the area inside of C, therefore the area inside of C plus the length of C' is equal

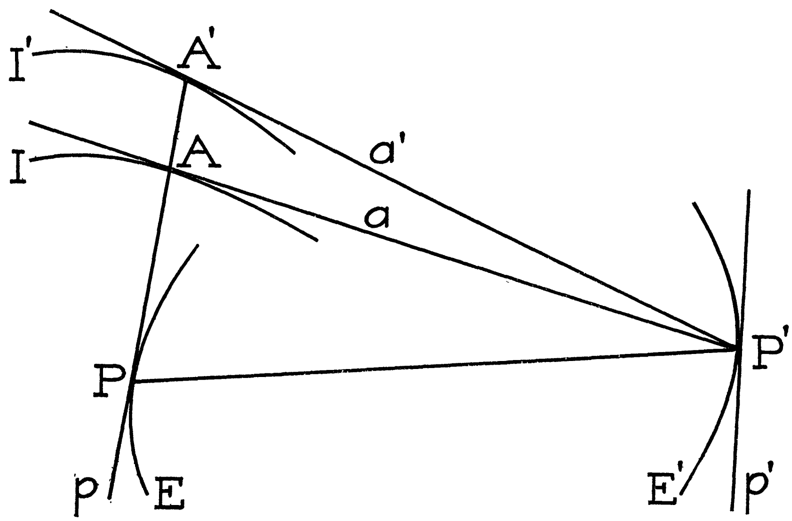
to 2π. So we have proved the known* theorem that the sum of the length of a closed spherical curve having no point-discontinuities and the area of its polar curve is equal to 2π.

Evolutes and Involutes.

7. We shall now consider the relation between spherical evolutes and involutes and its dual relation. In all that follows, in order to avoid circumlocution, we shall not usually distinguish between two diametrically opposite points or curves. We define the evolute of a curve as the envelope of its normals; the more precise definition and its dual are as follows:

The envelope of a great circle p passing though a variable point A of a curve I and making an angle $\frac{\pi}{2}$ with the tangent a at that point is a curve E called the *evolute* of I; I is then an *involute* of E.

The locus of a point P' on a variable tangent a' to a curve I' and situated at a distance $\frac{\pi}{2}$ from the point of tangency A' is a curve E' called the *polar evolute* of I'; I' is then a *polar involute* of E'.



In this figure the straight lines represent great circles; E and E' are polar curves; PP', of length $\frac{\pi}{2}$, is normal to both; p' and p are the polars of P and P', respectively.

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* Cf. Böklen, *Grunert's Archiv*, Vol. XLIII (1865), p. 18, and Maxwell's "Electricity and Magnetism," 3rd ed. (1892), Vol. II, p. 40.

It is evident that if I coincides with I', E and E' are polar curves; that is, the polar evolute of a curve is the polar of its evolute; whence the name polar evolute. Similarly, every polar involute of a curve is an involute of its polar (not the polar of an involute).

8. The relation between parallel curves (geodetic parallels on the sphere), on the other hand, is a self-dual one. Thus if I and I' are parallel curves (see figure), on which A and A' are corresponding points, and a and a' are the tangents at these points, respectively, then the great circle AA' is normal to both curves and the distance between A and A' is constant; dually, the point aa' ($= P'$) is at a distance $\frac{\pi}{2}$ from both A and A', and the angle between a and a' is constant.

Every curve has a singly infinite number of parallel curves. In a family of parallel curves the polar of every curve of the family is also included in the family. Every curve has an infinite number of involutes forming a family of parallel curves, and also an infinite number of polar involutes forming another family of parallel curves. Conversely, all the members of a family of parallel curves have the same evolute and the same polar evolute. Hence, any two polar curves have the same evolute and the same polar evolute.

9. The following dual theorems are evident from the figure:

The evolute E of a curve I is the locus of the center P of the circle of curvature* at a variable point A of I.	The polar evolute E' of a curve I' is the envelope of the axis p' of the circle of curvature at a variable point A' of I'.
On a tangent p to a curve E any two of its involutes I and I' cut off a segment AA' of constant length (independent of the position of the tangent).	The angle between the tangents a and a' that can be drawn from a point P' of a curve E' to any two of its polar involutes I and I' is constant (independent of the position of the point).

10. By means of the duality between arc and area we easily obtain the dual of the well-known theorem† that the length of an arc of the evolute is equal

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* The circle of curvature, or osculating circle, of a curve lying on a sphere is a circle of the sphere and can obviously be given a purely spherical definition. *E. g.*, see P. Serret, "Lignes à Double Courbure" (1860), pp. 29–35, where spherical evolutes are also considered, but not polar evolutes.

† See Serret, *loc. cit.*, p. 35.

to the difference between the corresponding radii of curvature of the involute. The following statement of the dual theorems is partly illustrated by the figure:

*If P and Q are points on a curve E and if the tangents at these points meet an involute I of the curve in the points A and B, respectively, then the length of the arc PQ of E is equal to the difference between the distances PA and QB.	*If p' and q' are tangents to a curve E' and if through their respective points of contact great circles a' and b' are drawn touching a polar involute I' of the curve, then the area exterior to E' between p' and q' is equal to the difference between the areas of the lunes $(p'a')$ and $(q'b')$. In other words, the angular displacement of a tangent to E' in moving from p' to q' is equal to the difference between the angles $(p'a')$ and $(q'b')$.

11. Since the semicircle extending along a common tangent a to two diametrically opposite curves I and $\bar{I}$ from one point of contact A to the other is bisected by the corresponding point P' of their polar evolute E', therefore the area between I and $\bar{I}$, bounded by two common tangents, is divided into two equal parts by E'.† This fact makes it possible to state the theorem of § 5 in the following form:

The area included between a curve I and its polar evolute and bounded by two great circles a_1 and a_2, which are tangents to I, is equal to the length of the corresponding arc of its polar curve bounded by the points A_1 and A_2, the poles of a_1 and a_2, respectively. Hence the duality:

The area between a curve and its polar evolute and bounded by two of its tangents.	The length of the arc of a curve bounded by two of its points.

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* A more precise statement of the limits within which these theorems are applicable is easily inferred from the analogous statement for the geometry of the plane given by Mangoldt in the *Encyclopädie der Mathematischen Wissenschaften*, Vol. III 3, p. 35.

† If E' is a closed curve, it evidently also divides the entire area of the sphere into two equal parts. See Serret, *loc. cit.*, pp. 90–91, where this theorem is ascribed to Jacobi.

Continuous Spherical Movements.

12. The kinematics of the sphere, or what is the same thing, the kinematics of the bundle whose vertex is the center of the sphere, deals with the movements or displacements which leave the sphere as a whole fixed, or which have one fixed point, the center of the sphere. We shall consider a continuous movement of the sphere; that is, the limit of the sum of an infinite number of infinitesimal rotations.

If throughout the entire movement a point P of the spherical surface (and its opposite point) remains fixed, then the movement becomes a simple rotation about P. Since p, the polar of P, also remains fixed, the movement can be regarded as a translation along p, provided a translation is defined as a spherical movement which leaves a great circle fixed, by analogy with a plane translation, which leaves a straight line fixed.

A *rotation* about a point P through an angle α; P is called the center of the rotation.	A *translation* along a great circle p through a distance α; p is called the axis of the translation.

Notice that a rotation and a translation are here the same movement considered from dual standpoints, and not, as in the kinematics of the plane, essentially distinct movements.

13. In the general case of any continuous spherical movement it is well-known[*] that the movement can be regarded as due to the rolling (without sliding) of a movable curve C_1 on a fixed curve C, and that C_1 and C are uniquely determined by the movement; at any instant the point of contact P of the two curves is the instantaneous center of rotation.

Let C_1' and C' be the polar curves of C_1 and C, respectively, and let p' be the polar of the point P; then C_1' and C' touch each other and their common tangent at the point of contact is p'. The instantaneous movement evidently consists of a translation along p'; and the entire movement may be regarded as due to a sliding of the movable curve C_1' on the fixed curve C', namely, the limit of the sum of an infinite number of infinitesimal translations. At any instant the common tangent p' is the instantaneous axis of translation.

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[*] Cf. Ziwet's "Theoretical Mechanics" (1893), Vol. I, p. 19, where the theorem is stated substantially as follows: Any continuous movement about a fixed point O can be produced by the rolling of one cone on another, both cones having their vertices at O. See also Koenigs, "Leçons de Cinématique" (1875), p. 187, theorems 3 and 4.

Consequently, the most general continuous movement of the sphere can be produced by the sliding (without rolling) of a movable curve C_1' on a fixed curve C'. C_1' and C' are uniquely determined by the movement.

Notice that a corresponding theorem does not hold in the kinematics of the plane, for there a sliding or translatory movement is of a very special character.

A movable curve C_1 *rolling* on a fixed curve C; their common point of contact P is the instantaneous *center of rotation.*	A movable curve C_1' *sliding* on a fixed curve C'; their common tangent p' at the point of contact is the instantaneous *axis of translation.*
If one of the two curves degenerates into a point, the other does, and the movement becomes a simple rotation about that point.	If one of the two curves degenerates into a great circle, the other does, and the movement becomes a simple translation along that great circle.
One of the two curves may degenerate into a great circle, but not both.	One of the two curves may degenerate into a point, but not both.

This special case will be considered in §§ 16–20.

14. In all that follows we shall assume the movement to be of such a nature that the curves C_1 and C, one of which rolls on the other, have no point-discontinuities, and therefore the polar curves C_1' and C', one of which slides on the other, have no tangential discontinuities. We shall now apply to these four curves the duality between arc and area.

It is intuitively evident that when one curve rolls on another, the lengths of the corresponding arcs of the two curves are equal. This fact and the one dual to it may be made explicit as follows:

If a curve C_1 rolls on a fixed curve C, and if to the points P and Q of C correspond, respectively, the points P_1 and Q_1 of C_1, then the length of the arc $P_1 Q_1$ is equal to the length of the arc PQ.	If a curve C_1' slides on a fixed curve C', and if to p' and q', tangents to C', correspond, respectively, p_1' and q_1', tangents to C_1', then the area exterior to C_1' between p_1' and q_1' is equal to the area exterior to C' between p' and q'.

Corresponding points and corresponding tangents are, of course, those which at some instant of the movement become common points of contact and common tangents at those points, respectively.

15. It is also well-known* that a continuous movement of the sphere can be represented by the path-curves of two of its points.

Therefore, a continuous movement of the sphere can be represented by the envelopes of two of its great circles.

Moreover, a continuous movement can be represented self-dually by the path-curve I_1 of a moving point A and the envelope I_2 of a moving great circle a.

Line-Element Movements.

16. In particular, A and a can always be chosen with a passing through A. If so, it may happen, in a certain type of movement, that I_1 and I_2 coincide; that is, that a fixed curve I exists which is at the same time the locus of A and the envelope of a, and moreover that throughout the movement the tangent to I at the point A is precisely the great circle a. This means that the line-element† (A, a) of the curve I is carried along the curve, remaining a line-element of I throughout the movement. We shall call such a movement a *line-element movement* and I a line-element curve of the movement.

17. The analogous case in the kinematics of the plane has been considered by Schoenflies‡ and the conclusion he draws holds also for the sphere. Namely, a line-element movement along the curve I (figure, § 7) is precisely the same as a movement produced by rolling its normal great circle p on its evolute E. Thus the instantaneous center of rotation is always at the center of curvature P of I.

This agrees with the usual derivation of the involute I from its evolute E; for the unwinding of a string amounts to the same thing as the rolling of a great circle.

By the principle of duality, now, we draw a further conclusion which does not hold for the plane. Namely, a line-element movement along I is the same as the movement produced by sliding the point P' on the polar evolute E'. It is to be observed that the sliding of P' on E' is a determinate movement, whereas the mere fact that E' is a path-curve of P' does not fix the movement.

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* Cf. Schoenflies, "Geometrie der Bewegung" (1886), Chap. 1, § 4, Art. 1, and Chap. 2, § 3, Art. 1.

† Cf. Lie-Scheffers, "Geometrie der Berührungstransformationen" (1896), p. 11.

‡ *Loc. cit.*, Chap. 1, § 6, Art. 5.

18. These results may be summed up as follows: Every line-element movement, by which a line-element (A, a) is carried along a curve I (figure, § 7), is identical with

the rolling of a great circle p on a fixed curve E; E is the evolute of I, and p is a normal to I.	the sliding of a point P' on a fixed curve E'; E' is the polar evolute of I, and P' is a point lying on a tangent to I at a distance $\frac{\pi}{2}$ from the point of tangency.
When a great circle p rolls on a curve E, the locus of every point A of p is an involute I of E and is also a line-element curve of the movement.	When a point P' slides on a curve E', the envelope of every great circle a passing through P' is a polar involute I of E', and is also a line-element curve of the movement.
The tangent to the involute I at the point A remains perpendicular to the rolling great circle p.	The point of tangency of the great circle a with the envelope I remains at a distance $\frac{\pi}{2}$ from the sliding point P'.
Every involute I of E is the locus of a point A lying on the rolling great circle p.	Every polar involute I of E' is the envelope of a great circle a passing through the sliding point P'.

Every line-element movement has an infinite number of line-element curves, which form a family of parallel curves.

19. As corollaries to the dual theorems of § 14 we have the following:

If a great circle p rolls on a fixed curve E, and if to the points P and Q of E correspond, respectively, the points P_1 and Q_1 of p, then the distance P_1Q_1 is equal to the length of the arc PQ.	If a point P' slides on a fixed curve E', and if to p' and q', tangents to E', correspond, respectively, p'_1 and q'_1, great circles passing through P', then the area of the lune $(p'_1\, q'_1)$ is equal to the area exterior to E' between p' and q'. In other words, the angle $(p'_1\, q'_1)$ is equal to the angular displacement of a tangent to E' in moving from p' to q'.

This dual result is also a special case of that obtained in § 10.

Inverse Line-Element Movements.

20. Any spherical movement can be regarded as a displacement of a movable sphere S' with respect to a fixed concentric sphere S. To an observer on S', however, S becomes movable and S' fixed; the resulting displacement of S with respect to S' is called the inverse movement.*

In the inverse movement the rôles of fixed and moving curves described in § 13 are interchanged; C rolls on C_1 and C' slides on C'_1.

It is to be noticed that the inverse movement is not in general the same as the original movement taken backwards. Thus in the original movement let C_1 roll on C from the position C_{10} to the position C_{11}; then in the backward movement C_1 will roll on C from the position C_{11} to the position C_{10}, while in the inverse movement C will roll on C_1. If, however, the original movement is a simple rotation, then the backward movement and the inverse movement are evidently identical.

An inverse line-element movement is therefore the same as the rolling of a curve E on a fixed great circle p, or dually the sliding of a curve E' through a fixed point P'. In the figure (§ 7) P will move along p and p' will revolve around P', as the curves E and E' roll and slide, respectively. Moreover, the line-elements (A, a), (A', a'), etc., infinite in number, whose points lie on the fixed great circle p and whose great circles pass through the fixed point P', will remain fixed, while the curves I, I', etc., will move in such a way as to continue to have these line-elements; *e. g.*, I will continue to touch the fixed great circle a at the fixed point A.

Closed Curves.

21. By a *closed curve* we shall mean one that is dually closed; that is, not only the locus of a continuously moving point which returns to its original position, but also the envelope of a continuously turning great circle which returns to its original position.

If two movements carry a sphere from the same initial position to the same final position, we shall call them *equivalent* movements. Every movement is therefore equivalent to a simple rotation about a point (translation along a great circle).

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* Cf. Schell, "Theorie der Bewegung und der Kräfte" (1879), Vol. I, p. 224; Schoenflies, *loc. cit.*, Chap. 1, § 2, Arts. 3–5, and "Encyclopädie," IV 1, p. 198. The *dualism* between a movement and its inverse, which is considered by Schell and Schoenflies, must not be confused with the *duality* which I am considering in this paper.

Let us now take up the case in which one of the two curves C_1 or C (and therefore also its polar curve C'_1 or C') is a closed curve, and in which the movement makes one or more complete circuits of this curve.

First suppose the moving curve to be closed.

If a closed curve C_1, of length α, rolls on a fixed curve C, and if in its initial and final positions the same point P_1 of C_1 coincides, respectively, with the points P and Q of C, then the movement is equivalent to a line-element movement along C from P to Q. If the point of tangency makes n complete circuits of C_1 from P_1 back to P_1, then the length of the arc PQ of C is equal to $n\alpha$.

If a closed curve C'_1, of exterior area α', slides on a fixed curve C', and if in its initial and final positions the same great circle p'_1, tangent to C'_1, coincides, respectively, with the great circles p' and q', tangents to C', then the movement is equivalent to a line-element movement along C' from p' to q'. If the common tangent makes n complete circuits of C'_1 from p'_1 back to p'_1, then the area exterior to C' between p' and q' is equal to $n\alpha'$.

22. Next consider the inverse movement, in which the fixed curve is closed.

If a curve C rolls on a fixed closed curve C_1, of length α, and if in its initial and final positions the points P and Q of C coincide, respectively, with the same point P_1 of C_1, then (letting p_1 be the tangent to C_1 at P_1) the movement is equivalent to an inverse line-element movement, in which C moves along the fixed line-element (P_1, p_1) from P to Q.

If a curve C' slides on a fixed closed curve C'_1, of exterior area α', and if in its initial and final positions the great circles p' and q', tangents to C', coincide, respectively, with the same great circle p'_1, tangent to C'_1, then (letting P'_1 be the point of contact of p'_1 with C'_1) the movement is equivalent to an inverse line-element movement, in which C' moves along the fixed line-element (P'_1, p'_1) from p' to q'.

A further conclusion concerning arcs and areas can be drawn here, exactly as in the last section.

23. The general theorems just enunciated (§§ 21 and 22) can now be applied to the case in which C is a great circle and C' a point, while C_1 and C'_1 are any

closed polar curves. The original movement is then a special kind of line-element movement or its inverse.

Thus the dual theorems of § 21, in this special case, take the following form (figure, § 7):

Every inverse line-element movement due to the rolling of a closed curve E, of length α, on a great circle p, and such that p makes a complete circuit of E, is equivalent to a simple translation along p through a distance α.	Every inverse line-element movement due to the sliding of a closed curve E', of exterior area $2\alpha'$, through a point P', and such that P' makes a complete circuit of E', is equivalent to a simple rotation about P' through an angle α'.

24. Similarly the theorems of § 22 take the following form (figure, § 7):

Every line-element movement produced by rolling a great circle once around a closed curve E, of length α, from the position p to the same position again is equivalent to a simple translation along p backward through a distance α.	Every line-element movement produced by sliding a point once around a closed curve E', of exterior area $2\alpha'$, from the position P' to the same position again is equivalent to a simple rotation about P' backward through an angle α', or forward through an angle $2\pi-\alpha'$ equal to the interior area of the curve.

The movement is of *finite period*, if and only if

the length of the curve	the area of the curve

is a rational multiple of π. In particular, the movement is of period 2 or equivalent to a

translation through the distance π, if the length of the curve is an odd multiple of *half the length of a great circle;*	rotation through the angle π, if the exterior area of the curve is an odd multiple of *the area of a hemisphere;*

and the movement is equivalent to the *identity*; that is, will carry every point of the sphere into itself,

if the length of the curve is an even multiple of half the length of a great circle.	if the exterior area of the curve is an even multiple of the area of a hemisphere.

25. In the special case in which E and E' are spherical triangles, the movement just considered becomes a succession of three simple rotations (translations); and if these triangles are assumed to have their sides and interior angles all > 0 and $< \pi$, the last theorem takes the following form:

A spherical movement that consists of three successive rotations about the vertices A, B, C of a spherical triangle through angles equal respectively to the three corresponding exterior angles of the triangle, and that causes a great circle starting at CA to *roll* once around the triangle, is equivalent to a simple translation along CA backwards through a distance equal to the perimeter of the triangle.	A spherical movement that consists of three successive translations along the sides a, b, c of a spherical triangle through distances equal respectively to the lengths of those sides, and that causes a point starting at the vertex ca to *slide* once around the triangle, is equivalent to a simple rotation about ca backwards through an angle equal to the sum of the exterior angles of the triangle, or forward through an angle equal to the *spherical excess*, that is, the interior area, of the triangle.*
The movement can not be of period 1, but will be of period 2, if the perimeter of the triangle is equal to two *quadrants* of a great circle, and will be of period 4, if the perimeter of the triangle is equal to one or three quadrants of a great circle.	The movement can not be of period 1, but will be of period 2, if the interior area of the triangle is equal to the area of two *octants* of the sphere, and will be of period 4, if the interior area of the triangle is equal to the area of one or three octants of the sphere.

26. In the figure of § 7 let E be a closed curve of length α having no cusps, and therefore E' a closed curve of exterior area 2α having no inflections; also suppose $AA' = \alpha$. Then the line-element movement by which p makes one complete circuit of E and P' one circuit of E' will carry A around into A',

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*By a curious chance this theorem, which was the starting-point of the investigation contained in this paper, first came to me as the result of a roundabout non-Euclidean line of argument. Afterwards a direct proof, without the use of duality, was communicated to me by Professor J. L. Coolidge, of Harvard University. Finally, I saw how obvious the dual theorem was.

so that I and I' must be parts of the same involute of E. Further, this involute will itself be a closed curve, if and only if α is a rational multiple of π. If E has cusps, and therefore E' has inflections, this theorem will still hold good, provided the length of E and the exterior area of E' are determined according to the convention laid down in § 5.

Hence a necessary and sufficient condition

that the involutes of a closed curve are also closed curves is that its length be a rational multiple of π.	that the polar involutes of a closed curve are also closed curves is that its area be a rational multiple of π.

Application to the Riemannian Plane.

27. In Riemannian space there are four simple two-dimensional configurations whose geometries are closely analogous to that of the Euclidean (or Riemannian) sphere, namely the plane, the bundle or sheaf, the right paratactic congruence, and the left paratactic congruence, paratactic lines being Clifford parallels.* All the dual theorems of this paper are immediately applicable to each of the four geometries. At present I merely wish to consider briefly the first one, that of the plane.

If the Riemannian space is double, the plane is so perfect a picture of the sphere that the only change needed is to read "straight line" for "great circle." We assume, of course, that the radius of curvature of the space is $+1$.

28. If the space is single, a few other changes will be needed and the duality will become simpler in some respects than before. Thus, diametrically opposite points and curves become coincident points and curves, and the point becomes dual to the undirected straight line; in § 4, instead of having "the area swept out by a great circle" defined in a rather artificial way, we see that the new phrase "the area swept out by a straight line" will mean exactly what it says, since the entire length of the straight line is π.

It is to be noticed that if a tangent to a closed curve in the single Riemannian plane makes one complete circuit of the curve, it will sweep out the entire exterior area of the curve just twice, and therefore that while the area swept out is *twice* the entire length of the polar curve, the exterior area of the original curve is exactly *equal* to the length of the polar curve. On this account it will be necessary in the right-hand theorems of §§ 21 and 22 to replace "exterior area α'" by "exterior area $\frac{\alpha'}{2}$," and similarly in §§ 23 and 24 to replace "exterior area $2\alpha'$" by "exterior area α'." Moreover, in the right-hand theorem of § 24 "the area of a hemisphere" must be replaced by "half the area of a plane," and in the left-hand theorem of the same section "half the length of a great circle" must be replaced by "the length of a straight line."

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*I have made a study of the relation between these four geometries in a paper which I am preparing for publication elsewhere.

When the suggested alterations and the obvious changes of language that go with them have been made, the results of this paper will take their place as a set of dual theorems in the geometry of the single Riemannian plane.

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